Computation of Self-Similar Solution Profiles for the Nonlinear Schrödinger Equation

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We consider the classical nonlinear Schrödinger equation in three space dimensions,

$$\mathrm{i}\frac{\partial u}{\partial t} + \Delta u + |u|^2 u = 0, \quad t > 0, \tag{1}$$

which occurs in various important applications for example in nonlinear optics or plasma physics. (1) has solutions that become unbounded in finite time. This occurs at a single point at which there is a growing and increasingly narrow peak. In this case, it is conjectured that the solutions blow up in a self-similar way [3].

The problem of the computation of this self-similar solution profile reduces to a nonlinear, ordinary differential equation on an unbounded domain, where the boundary conditions are carefully chosen at infinity to avoid rapidly oscillating solutions. We show that a transformation of the independent variable to the interval $[0, 1]$ yields a well-posed boundary value problem with an essential singularity. This can be stably solved by polynomial collocation [2]. Moreover, our MATLAB solver `sbvp` [1] can be applied to solve the problem efficiently and provides a reliable estimate of the global error of the collocation solution. This is possible because the boundary conditions for the transformed problem serve to eliminate undesired, rapidly oscillating solution modes and essentially reduce the problem of the computation of the physical solution of the problem to a boundary value problem with a singularity of the first kind, see [4].

Keywords: Nonlinear Schrödinger equation, self-similarity, blow-up solutions, essential singularity, collocation methods, error estimation.

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