Acceleration Techniques for Singular Initial Value Problems

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Abstract: The acceleration technique known as Iterated Defect Correction (IDeC) for the numerical solution of singular initial value problems is investigated. IDeC based on the implicit Euler method performs satisfactorily and can thus be used for the efficient solution of singular boundary value problems with the shooting method. Higher order one-step methods like the box scheme or the trapezoidal rule cannot serve as a basic method because of a break-down of the asymptotic expansions of the global error caused by the singularity. The theoretical considerations are also supported by a comparison with extrapolation methods. Finally, it is shown that for similar reasons IDeC cannot be used for *singular terminal value problems*.

Key-Words: singular initial value problems, Iterated Defect Correction, asymptotic error expansions.

1 Introduction

We consider the numerical solution of *singular initial value problems* of the form[1]

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \tag{1a}$$
$$B_0 z(0) = \beta, \tag{1b}$$
$$z \in C[0,1], \tag{1c}$$

where z and f are n-dimensional vector-valued functions, M is an $n \times n$ matrix, and $B_0 \in \mathbb{R}^{r \times n}$ and $\beta \in \mathbb{R}^r$ are chosen in such a way that problem (1) is well-posed. More precisely, in [10] and [9] it was shown under mild smoothness assumptions on f and M that a restriction on the spectrum of $M(0)$, namely the absence of purely imaginary eigenvalues or eigenvalues with positive real parts, is necessary in order to formulate an initial value problem of the form (1) having a unique, continuous solution $z(t)$. In this case, the condition $M(0)z(0) = 0$ is necessary and sufficient for $z \in C[0,1]$ and provides $n-r$ linearly independent conditions which the initial value $z(0)$ has to satisfy. Here, r is the dimension of the kernel of $M(0)$. This solution is unique iff the $r \times r$ matrix $B_0 \tilde{E}$ is nonsingular, where $\tilde{E}$ is the basis of the kernel of $M(0)$. If f is k times continuously differentiable and $M \in C^{k+1}[0,1]$, then the solution satisfies $z \in C^{k+1}[0,1]$.

Moreover, *singular terminal value problems* are investigated,

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \tag{2a}$$
$$B_1 z(1) = \beta, \tag{2b}$$
$$z \in C[0,1], \tag{2c}$$

where again the conditions (2b) yield a well-posed problem. It was shown in [9] that in this case as well a special structure of the spectrum of $M(0)$ is necessary and sufficient for (2) to have a continuous solution[2]. Thus, n linearly independent conditions (2b) are required to obtain a unique solution. This solution satisfies $z \in C^{k+1}[0,1]$ if f is k times continuously differentiable, $M \in C^{k+1}[0,1]$ and the smallest positive real part of the eigenvalues of $M(0)$ is greater than $k+1$.

Singular ordinary differential equations with boundary conditions posed at one point arise for example in the context of the solution of two-point boundary value problems by shooting methods. Singular two-point boundary value problems in turn often describe symmetric solutions of partial differential equations from applications in physics

[1] This type of singularity is called a *singularity of the first kind*.

[2] We reqire the absence of purely imaginary eigenvalues, eigenvalues with negative real parts or a multiple eigenvalue 0, where the associated block in the Jordan canonical form is not diagonal.

(see for example [3]), chemistry (cf. [14]) or mechanics (buckling of spherical shells, [4]). Moreover, there are some models in ecology (avalanche run-out, [13]) that are posed as singular initial value problems of the form (1).
For the numerical solution of (1) various schemes were proposed. It turns out, however, that many high-order methods show order reductions when applied to singular problems. Explicit Runge-Kutta methods for example show a reduction down to order 2 in general, see [7], and multi-step methods deviate from their classical convergence order by a logarithmic term, cf. [6].
Another approach to obtain a high order solution is to use an acceleration technique: problems (1) and (2) are first solved with a basic method of low order and then a suitable iteration procedure is used to enhance the order of the approximation. To this aim we consider the *Iterated Defect Correction (IDeC)* described in §2. The performance of IDeC is compared with the extrapolation method.

2 Iterated Defect Correction

For the numerical treatment of (1) we write the admissible intitial condition (1b) in the form $z(0) = \tilde{\beta}$. Moreover, we assume to know the approximate solution $z_h^{[0]} := z_h = (z_0, \dots, z_N)$ obtained by some discretization method φ_h on a grid $\Delta_h := (t_0, \dots, t_N)$, $t_i = ih$, $h = \frac{1}{N}$, and denote by $p^{[0]}(t)$ the polynomial of degree N interpolating the values of $z_h^{[0]}$. Using this interpolating function, we construct a neighboring problem associated with (1) and solved exactly by $p^{[0]}(t)$,

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)) + d^{[0]}(t), \quad (3a)$$
$$z(0) = p^{[0]}(0) = \tilde{\beta}, \quad (3b)$$

where

$$d^{[0]}(t) := p^{[0]\prime}(t) - \frac{M(t)}{t} p^{[0]}(t) - f(t, p^{[0]}(t)).$$

We now solve (3) by the same numerical method φ_h and obtain an approximate solution $p_h^{[0]}$ for $p^{[0]}(t)$. This means that for the solution of the neighboring problem (3) we know the global error which we can use to estimate the unknown error of the original problem (1) and use this information to improve the solution[3],

$$z_h^{[1]} := z_h^{[0]} + \left(R_h \left(p^{[0]} \right) - p_h^{[0]} \right).$$

We use these values to define a new interpolating polynomial $p^{[1]}(t)$ by requiring $p^{[1]}(t_j) = z_j^{[1]}$, $j = 0, \dots, N$. Now $p^{[1]}(t)$ is used to define a neighboring problem in the same manner as for (3), where again the exact solution is known, and the numerical solution of this neighboring problem serves to obtain the second improved solution $z_h^{[2]} := z_h^{[0]} + \left(R_h \left(p^{[1]} \right) - p_h^{[1]} \right)$. Clearly, this procedure can be iteratively continued.
For obvious reasons one does not use one interpolating polynomial for the whole interval in practice. Instead, a piecewise polynomial function composed of polynomials of (moderate) degree m is defined to specify the neighboring problem. Due to classical theory, see [5], this parameter constitutes a bound for the level of accuracy which can be achieved by the iteration described above. This will be explained in more detail in the following sections.

3 The Implicit Euler Method

If the implicit Euler method is used as basic method for the IDeC iteration for the solution of (1), the classical order sequence (cf. [5]) for the respective iterates can be observed. This was observed experimentally in [1] and [10], and only recently proven theoretically in [12]. The main result of this paper is given in the following
Theorem: *Consider the IDeC method based on the implicit Euler rule and on piecewise interpolation with polynomials of degree m for the numerical solution of problem* (1). *For the approximations obtained in the course of the iteration,*

$$\| z_h^{[j]} - R_h(z) \|_h := \max_{0 \le l \le N} | z_l^{[j]} - z(t_l) | = O(h^{j+1}) \quad (4)$$

holds for $j = 0, \dots, m-1$, provided that f and M are sufficiently smooth. In this case (polynomials of degree m are used for the interpolation), further iteration does not increase the asymptotic order of the approximation in general.
The main idea of the proof is to use an asymptotic expansion of the global error of the implicit Euler

[3] $R_h(z) := (z(t_0), \dots, z(t_N))$ for a continuous function z.

method. For sufficiently smooth data f and M we can prove the existence of an expansion of the form

$$z_h - R_h(z) = \sum_{j=1}^{m} h^j R_h(e_j) + r_h, \tag{5}$$

for any m, where $e_j,\ j = 1, \dots, m$, are smooth functions defined by the *variational equations*, singular initial value problems which are linearized versions of (1) with homogeneous initial conditions. Moreover, $\|r_h\|_h = O(h^{m+1})$. Under the same assumptions, similar expansions exist for the neighboring problems (3) and equally for $p_h^{[l]},\ l = 1, \dots, m-2$. Since the neighboring problems depend on h, so do the associated error expansions. Written in terms of a step-size $\bar{h} = \frac{h}{\nu},\ \nu \in \mathbb{N}$, they read

$$p_{\bar{h}}^{[l]} - R_{\bar{h}}\left(p^{[l]}\right) = \sum_{j=1}^{m} \bar{h}^j R_{\bar{h}}\left(e_j^{[l]}\right) + r_{\bar{h}}^{[l]}. \tag{6}$$

The functions $e_j^{[l]}$ may have jump discontinuities in the first derivatives at the switch points between the polynomials defining the neighboring problems, but in the interior of these intervals they have the same smoothness properties as e_j from (5). For the remainder term, a similar estimate holds which is independent of h.

Let $\bar{h} := h$. Then from the existence of the error expansions for the original and the neighboring problems we can conclude for $l = 0, \dots, m-2$

$$R_h(z) - z_h^{[l+1]} = \sum_{j=1}^{m} h^j \left(R_h(e_j) - R_h\left(e_j^{[l]}\right)\right) + r_h - r_h^{[l]}. \tag{7}$$

Consequently, the proof is completed by showing that

$$\|R_h(e_j) - R_h\left(e_j^{[l]}\right)\|_h = O(h^{2+l-j}), \tag{8}$$

holds for $j = 1, \dots, m,\ l = 0, \dots, m-2$. For the technical details of the proof we refer to [12].

Thus, for smooth data f and M, the IDeC method based on the implicit Euler scheme can potentially yield approximations to singular initial value problems of an arbitrary order. This fact was for example used successfully in a shooting code for singular boundary value problems[4]. For numerical results of this effort see [1], or [11] for a theoretical investigation of shooting methods.

The existence of an error expansion (5) also makes the use of extrapolation methods possible. Thus, the classical order sequence $O(h),\ O(h^2), \dots$ could be observed for Richardson extrapolation based on the implicit Euler method, cf. [2].

It seems natural however to improve the efficiency of the IDeC method by using a higher order scheme like the box scheme or the trapezoidal rule as basic method, where for regular problems order sequences $O(h^2),\ O(h^4), \dots$ are observed, or alternatively to use the (computationally cheaper) explicit Euler method. It is the aim of the next sections to show why neither of these is possible for singular initial value problems.

4 The Box Scheme

A proof of convergence of the box scheme applied to (1) was given in [8]. It turns out that it retains its classical convergence order $O(h^2)$ except for the case where a multiple eigenvalue 0 of $M(0)$ occurs. Here, the convergence order is $O(|\ln(h)|^{n_0-1}h^2)$, where n_0 denotes the dimension of the largest Jordan block associated with the eigenvalue 0. Clearly, the numerical solution z_h cannot have an asymptotic error expansion (5). The IDeC method doesn't work for these problems in general, see the numerical experiments in [2]. But even when the basic solution shows its classical convergence order, IDeC need not work successfully. This can be seen from the numerical results for the test problem (9) in Table 1, which lists maximal absolute error, empirical rate of convergence and error constant for the basic solution and the first IDeC iterate, respectively. Further iteration does not increase the order of accuracy. The reason for this failure is the break-down of the expansion (5) for the problem

$$z'(t) = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix} \cdot z(t) + \mathrm{e}^{2t} t \begin{pmatrix} 0 \\ 4t^2 + 26t + 35 \end{pmatrix}, \tag{9a}$$

$$z(0) = (0, 0), \tag{9b}$$

with exact solution $z(t) = (t^2\mathrm{e}^{2t}, 2(t+1)t^2\mathrm{e}^{2t}) \in C^\infty[0,1]$. The coefficient functions can easily be

[4]We have to restrict ourselves to BVPs where the IVPs associated with shooting are well-defined.

h	δ_0	p_0	c_0	δ_1	p_1	c_1
$1/5 \cdot 2^{-3}$	$4.0 \cdot 10^{-02}$	1.999	$-6.3 \cdot 10^{+01}$	$2.6 \cdot 10^{-03}$	1.996	$-4.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$1.0 \cdot 10^{-02}$	1.999	$-6.4 \cdot 10^{+01}$	$6.6 \cdot 10^{-04}$	1.999	$-4.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$2.5 \cdot 10^{-03}$	1.999	$-6.4 \cdot 10^{+01}$	$1.6 \cdot 10^{-04}$	1.999	$-4.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$6.2 \cdot 10^{-04}$	1.999	$-6.4 \cdot 10^{+01}$	$4.1 \cdot 10^{-05}$	1.999	$-4.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	$1.5 \cdot 10^{-04}$	1.999	$-6.4 \cdot 10^{+01}$	$1.0 \cdot 10^{-05}$	1.999	$-4.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-8}$	$3.9 \cdot 10^{-05}$	1.999	$-6.4 \cdot 10^{+01}$	$2.5 \cdot 10^{-06}$	1.999	$-4.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-9}$	$9.7 \cdot 10^{-06}$	1.999	$-6.4 \cdot 10^{+01}$	$6.4 \cdot 10^{-07}$	1.999	$-4.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-10}$	$2.4 \cdot 10^{-06}$	1.999	$-6.4 \cdot 10^{+01}$	$1.6 \cdot 10^{-07}$	1.999	$-4.2 \cdot 10^{+00}$

Table 1: IDeC with box scheme for (9)

computed from the associated variational equations. It turns out that

$$\begin{aligned} e_1(t) &= (0,0), \\ e_2(t) &= \Big(-\frac{c_1}{5} t^{-5} - \Big(\frac{c_2}{3}+1\Big) t^{-3} - 1 + O(t), \ldots \\ &\qquad c_1 t^{-5} + (c_2+3) t^{-3} - 2 + O(t) \Big), \end{aligned}$$

and there is no choice of the constants in the general solution for e_2 which would yield a continuous solution satisfying the homogeneous initial conditions. Thus, the asymptotic error expansion breaks down and IDeC does not improve the convergence orders of the iterates. The basic convergence order $O(h^2)$ is preserved nonetheless.

In [2] examples can be found where, due to special properties of the solution, the asymptotic error expansion breaks down at a later stage. This is reflected in the performance of IDeC, yielding order sequences $O(h^2)$, $O(h^4)$, $O(h^6)$, $O(h^6), \ldots$ for instance.

5 The Trapezoidal Rule

When analyzing the trapezoidal rule, we are faced with a different situation. The basic convergence order $O(h^2)$ could be shown for all well-posed problems (1) in [8][5]. In this case, however, even the existence of smooth coefficient functions e_j in (5) does not guarantee the existence of a smooth error expansion. This paradoxical situation is obviously caused by the fact that the remainder term in (5) cannot be estimated at the necessary level of accuracy. We confirm this fact by comparing the performance of IDeC with that of extrapolation. For regular problems, we expect to observe order sequences $O(h^2)$, $O(h^4), \ldots$ for both methods. When applied to (9), however, IDeC shows a sequence $O(h^2)$, $O(h^3)$ without further improvement after the first IDeC step, see [2]. This behavior is also reflected in the asymptotic properties of extrapolation as listed in Table 2. No better result than $O(h^3)$ could be achieved.

In contrast to this example, however, for many other test problems the classical order sequence can be observed for both acceleration methods, cf. [2].

6 The Explicit Euler Method

In the view of the failure of higher order methods, it seems natural to try the (computationally cheaper) explicit Euler method instead of the implicit Euler rule as a basis for IDeC. Although the classical convergence order of the explicit Euler, $O(h)$, holds (see [8]), the IDeC based on this method does not work satisfactorily for (1). Due to an apparent break-down of the asymptotic error expansion, for the majority of test problems no higher convergence order than 2 can be expected. As in §5 this is confirmed by the identical behavior of the extrapolation method. For a more detailed discussion see [2].

7 Terminal Value Problems

To explain the behavior of the IDeC method for (2), we restrict our attention to the explicit Euler rule. The behavior of the other methods is essentially similar[6].

The proof of the basic convergence order $O(h)$ of the explicit Euler method for (2) requires only slight modifications of the techniques for the implicit Eu-

[5]Note that an evaluation of the right-hand side at $t=0$ can be replaced by $z'(0)$, which is known in the case of (1).

[6]For implicit methods, however, there are additional difficulties due to the evaluation of the right-hand side at $t=0$.

h	δ_0	p_0	c_0	δ_1	p_1	c_1
$1/5 \cdot 2^{-2}$	$2.0 \cdot 10^{-02}$	1.999	$8.2 \cdot 10^{+00}$	$1.0 \cdot 10^{-04}$	2.974	$7.8 \cdot 10^{-01}$
$1/5 \cdot 2^{-3}$	$5.1 \cdot 10^{-03}$	2.000	$8.2 \cdot 10^{+00}$	$1.3 \cdot 10^{-05}$	2.994	$8.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-4}$	$1.2 \cdot 10^{-03}$	2.000	$8.2 \cdot 10^{+00}$	$1.6 \cdot 10^{-06}$	2.999	$8.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-5}$	$3.2 \cdot 10^{-04}$	2.000	$8.2 \cdot 10^{+00}$	$2.1 \cdot 10^{-07}$	3.000	$8.6 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$8.0 \cdot 10^{-05}$	2.000	$8.2 \cdot 10^{+00}$	$2.6 \cdot 10^{-08}$	3.000	$8.6 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$2.0 \cdot 10^{-05}$	2.000	$8.2 \cdot 10^{+00}$	$3.2 \cdot 10^{-09}$	3.000	$8.6 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$5.0 \cdot 10^{-06}$	2.000	$8.2 \cdot 10^{+00}$	$4.1 \cdot 10^{-10}$	3.000	$8.6 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$1.2 \cdot 10^{-06}$	2.000	$8.2 \cdot 10^{+00}$	$5.1 \cdot 10^{-11}$	3.000	$8.6 \cdot 10^{-01}$

Table 2: Extrapolation with trapezoidal rule for (9)

ler method for (1). Nonetheless, IDeC breaks down due to a counter-intuitive unsmoothness of the solutions of the variational equations. Recall that the smoothness of the solution of a problem of the form (2) depends not only on the smoothness of the data f and M, but also on the eigenvalues of $M(0)$. Thus, it may occur that even if a problem's solution is sufficiently smooth, this does not hold for the solutions of the associated variational equations. This is the case for the following test problem.

$$\begin{aligned} z'(t) &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot z(t) \\ &\quad + te^t \begin{pmatrix} 0 \\ t^2+4t+2 \end{pmatrix}, \qquad (10a) \\ z(1) &= (\mathrm{e}-1, 3\mathrm{e}+1), \qquad (10b) \end{aligned}$$

with exact solution $z(t) = (t^2\mathrm{e}^t + t - 2, (2+t)t^2\mathrm{e}^t + t) \in C^\infty[0,1]$. The coefficient functions in (5) are

$$\begin{aligned} e_1(t) &= \Big(\frac{7\mathrm{e}}{2} + 2t\ln(t), 2t\ln(t)\Big) + O(t), \\ e_2(t) &= \Big(-\frac{7}{6} + \frac{5\mathrm{e}}{3} + 6t\ln(t), \dots \\ &\quad -1 + 6t\ln(t)\Big) + O(t), \\ e_3(t) &= \Big(-\frac{1}{6}t^{-1} - 3 + 6t\ln(t), \dots \\ &\quad -\frac{1}{6}t^{-1} - 3 + 6t\ln(t)\Big) + O(t). \end{aligned}$$

From this observation, it is clear why the IDeC for this problem shows the order sequence $O(h)$, $O(h^2), \dots$ without further improvement, see Table 3.

The early break-down of the error expansion can be explained by the fact that the positive eigenvalue of $M(0)$ for (10) is equal to 1. If this value is larger, longer expansions can be derived for problems (2). Nonetheless, e_j become unsmooth eventually even in this case.

8 Conlusions

It turns out that Iterated Defect Correction can indeed be used to obtain high-order approximations for singular initial value problems (1). However, this is only the case when the implicit Euler method is used as basic method. Higher order methods like the box scheme or the trapezoidal rule and computationally cheaper methods like the explicit Euler method cannot be used to serve this purpose. The reason is the break-down of the expansions for their global error defined in (5), which we found to be crucial for the fast convergence of the IDeC, see the proof for the implicit Euler method in [12].

For terminal value problems (2), similar effects could be observed. In this case, however, the reason for the break-down of (5) seems to be an unsmoothness of the solutions of problems (2) related to the spectrum of the Matrix $M(0)$.

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h	δ_0	p_0	c_0	δ_1	p_1	c_1
$1/5 \cdot 2^{-4}$	$1.1 \cdot 10^{-01}$	1.002	$-9.6 \cdot 10^{+00}$	$2.1 \cdot 10^{-03}$	2.015	$-1.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-5}$	$5.9 \cdot 10^{-02}$	1.001	$-9.5 \cdot 10^{+00}$	$5.2 \cdot 10^{-04}$	2.007	$-1.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-6}$	$2.9 \cdot 10^{-02}$	1.000	$-9.5 \cdot 10^{+00}$	$1.2 \cdot 10^{-04}$	2.003	$-1.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-7}$	$1.4 \cdot 10^{-02}$	1.000	$-9.5 \cdot 10^{+00}$	$3.2 \cdot 10^{-05}$	2.001	$-1.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-8}$	$7.4 \cdot 10^{-03}$	1.000	$-9.5 \cdot 10^{+00}$	$8.0 \cdot 10^{-06}$	2.000	$-1.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-9}$	$3.7 \cdot 10^{-03}$	1.000	$-9.5 \cdot 10^{+00}$	$2.0 \cdot 10^{-06}$	2.000	$-1.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-10}$	$1.8 \cdot 10^{-03}$	1.000	$-9.5 \cdot 10^{+00}$	$5.0 \cdot 10^{-07}$	2.000	$-1.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-11}$	$9.2 \cdot 10^{-04}$	1.000	$-9.5 \cdot 10^{+00}$	$1.2 \cdot 10^{-07}$	2.000	$-1.3 \cdot 10^{+01}$

h	δ_2	p_2	c_2	δ_3	p_3	c_3
$1/5 \cdot 2^{-4}$	$4.7 \cdot 10^{-05}$	2.123	$-5.2 \cdot 10^{-01}$	$4.9 \cdot 10^{-05}$	2.049	$-3.9 \cdot 10^{-01}$
$1/5 \cdot 2^{-5}$	$1.0 \cdot 10^{-05}$	2.116	$-5.0 \cdot 10^{-01}$	$1.1 \cdot 10^{-05}$	2.025	$-3.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$2.5 \cdot 10^{-06}$	2.077	$-4.0 \cdot 10^{-01}$	$2.9 \cdot 10^{-06}$	2.012	$-3.2 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$5.9 \cdot 10^{-07}$	2.045	$-3.2 \cdot 10^{-01}$	$7.2 \cdot 10^{-07}$	2.006	$-3.0 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$1.4 \cdot 10^{-07}$	2.024	$-2.8 \cdot 10^{-01}$	$1.7 \cdot 10^{-07}$	2.003	$-3.0 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$3.5 \cdot 10^{-08}$	2.012	$-2.5 \cdot 10^{-01}$	$4.4 \cdot 10^{-08}$	2.001	$-2.9 \cdot 10^{-01}$
$1/5 \cdot 2^{-10}$	$8.8 \cdot 10^{-09}$	2.006	$-2.4 \cdot 10^{-01}$	$1.1 \cdot 10^{-08}$	2.000	$-2.9 \cdot 10^{-01}$
$1/5 \cdot 2^{-11}$	$2.1 \cdot 10^{-09}$	2.003	$-2.3 \cdot 10^{-01}$	$2.7 \cdot 10^{-09}$	2.000	$-2.9 \cdot 10^{-01}$

Table 3: IDeC with explicit Euler method for (10)

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