A proof and further details can be found in [7].

The full set of requirements on methods for nonlinear DAEs has to be taken into account when constructing practical methods. A test implementation Glimda [8] has been developed that employs General LInear Methods for Differential Algebraic equations. The code implements a variable-stepsize, variable-order approach, where methods of order 1,2 and 3 are used.

The preliminary code Glimda based on general linear methods seems to be competitive with BDF and Runge-Kutta solvers. By construction Glimda has advantages for MNA equations. Hence there is strong evidence that general linear methods can be used efficiently for solving differential algebraic equations in integrated circuit design.

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Collocation Methods for Index-1 DAEs with a critical point

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(joint work with O. Koch, R. März, D. Praetorius)

Model problem. We investigate the convergence behavior of collocation schemes applied to approximate solutions of index-1 DAEs, including the case when a critical point of $1-A$ type is present, see [6] and [5] for more technical details. The underlying analytical problem is the linear system of DAEs,

$$A(t)(D(t)x(t))' + B(t)x(t) = g(t), \quad t \in (0,1], \tag{1}$$

where $A(t) \in \mathbb{R}^{m\times n}$, $D(t) \in \mathbb{R}^{n\times m}$, $B(t) \in \mathbb{R}^{m\times m}$ and $g(t),\ x(t) \in \mathbb{R}^m$ with $n \le m$. We assume that $D(t) \equiv D$ is a constant matrix and that the matrices A, B and the inhomogeneity g are at least continuous, A, B, $g \in C[0,1]$.

Example: The following two dimensional problem belongs to class (1) and has a solution $x_1(t) = -\frac{6t+1}{2}e^{5t}$, $x_2(t) = -\frac{8t+1}{2}e^{5t}$:

$$\begin{pmatrix}1\\1\end{pmatrix}(1,-1)\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}' + \begin{pmatrix}2 & 0\\0 & t+2\end{pmatrix}x(t) \;=\; \begin{pmatrix}-te^{5t}\\-\frac{8t+7}{2}te^{5t}\end{pmatrix}. \tag{2}$$

We study systems (1) with properly stated leading term, cf. [1]. This means that A and D are well matched, i.e., $\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n$, $t \in (0,1]$, and there exists a projector function $R \in C^1(0,1]$ which realizes this splitting. Here, we assume that $\ker(A(t)) = \{0\}$, $t \in (0,1]$ and $\operatorname{im}(D) = \mathbb{R}^n$. Let Q_0 be a projector onto $N_0 := \ker(A(t)D) \equiv \ker(D)$ and let us define $P_0 := I - Q_0$. In our case, since the matrix D is constant, $R = I$ for $t \in (0,1]$, and Q_0, P_0 are constant, we regard all projections as extended to the interval $[0,1]$. In order to describe the boundary/initial conditions which are necessary and sufficient for (1) to be well-posed, we decouple the system using techniques from [1]. To this end, we introduce the matrices $G_0(t) := A(t)D$, $G_1(t) := G_0(t) + B(t)Q_0$ and allow a critical point at $t = 0$, where G_1 may become singular, i.e. $G_1(t)$ is non-singular on $(0,1]$. The decoupled system reads:

$$u'(t) + DG_1^{-1}(t)B(t)D^-u(t) = DG_1^{-1}(t)g(t), \quad t \in (0,1], \tag{3}$$

$$Q_0x(t) = -Q_0G_1^{-1}(t)B(t)D^-u(t) + Q_0G_1^{-1}(t)g(t), \quad t \in (0,1], \tag{4}$$

where $u(t) := Dx(t)$ are differential and $Q_0x(t)$ are algebraic components of the solution $x(t)$, and D^- is a reflexive generalized inverse of D. We now rewrite (3) and obtain a system of singular ODEs with a singularity of the first kind[1],

$$u'(t) - \frac{1}{t}M(t)u(t) = f(t), \;\; t \in (0,1], \tag{5}$$

where $M(t)/t := -DG_1^{-1}(t)B(t)D^-$, $f(t) := DG_1^{-1}(t)g(t)$. Let us assume that $M \in C^1[0,1]$ and $f \in C[0,1]$. Then we can use the theory given in [3] to augment (5) by a set of initial[2] conditions necessary and sufficient for $u \in C[0,1]$. In case that $M(0)$ has zero eigenvalues or eigenvalues with negative real parts, u needs to satisfy $u(0) = \gamma$, where $\gamma \in \ker M(0)$. Finally, if the right-hand side in (4) is continuous on $[0,1]$, then there exists a unique, continuous solution of the following IVP:

$$A(t)Dx'(t) + B(t)x(t) = g(t), \quad t \in (0,1], \tag{6}$$

$$Dx(0) = \gamma, \;\; Q_0x(0) = \lim_{t\to 0}(-Q_0G_1^{-1}(t)B(t)D^-\gamma + Q_0G_1^{-1}(t)g(t)) =: Q_0x_0. \tag{7}$$

Collocation scheme. We now turn to the numerical treatment of the IVP (6), (7). We first introduce a mesh $\Delta := (\tau_0, \tau_1, \dots, \tau_N)$, with $h_i := \tau_{i+1} - \tau_i$, $i = 0, \dots, N-1$, $\tau_0 = 0$, $\tau_N = 1$, such that $h_i \le h$. In each subinterval $J_i = [\tau_i, \tau_{i+1}]$, we place m distinct collocation points, $\tau_i < t_{i,j} < \tau_{i+1}$, $j = 1, \dots, m$. We approximate $x(t)$ by a function $p(t) = p_i(t)$, $t \in J_i$, where $p \in \mathbf{B}_m$, and $\mathbf{B}_m$ is

[1]Singularity of the first kind arises when we assume that $t = 0$ is an algebraically simple zero of the determinant of $G_1(t)$.

[2]We restrict our attention to IVPs in this talk.

the Banach space of globally continuous, piecewise polynomial functions of degree $\leq m$ equipped with the maximum norm. The defining equations for p are, $j = 1, \ldots, m,\ i = 0, \ldots, N-1$,

$$A(t_{i,j})Dp'(t_{i,j}) + B(t_{i,j})p(t_{i,j}) = g(t_{i,j}), \tag{8}$$

$$Dp(0) = \gamma, \quad Q_0 p(0) = Q_0 x_0. \tag{9}$$

Note, that the numerical method is applied to the IVP (6), (7) in its original form. We first show that $p \in \mathbf{B}_m$ exists and is unique. Decoupling (8) yields a collocation scheme for the differential components of p, $q(t) := Dp(t)$, and it follows from [4] that $q(t) \in \mathbf{B}_m$ exists and is unique. Then, it is easy to see that $Q_0 p(t) \in \mathbf{B}_m$ exists and is unique and consequently, this also holds for $p(t) \in \mathbf{B}_m$.

In order to derive the error bounds for the solution p, we introduce an error function $e \in \mathbf{B}_m$ defined by, $j = 1, \ldots, m,\ i = 0, \ldots, N-1$,

$$e'(t_{i,j}) = x'(t_{i,j}) - p'(t_{i,j}), \quad e(0) = 0. \tag{10}$$

Standard results for interpolation, see [2], yield the estimate for the interpolation error $e'(t) = x'(t) - p'(t) + P_0 O(h^k) + Q_0 O(h^l)$. Integrating this expression, we obtain $e(t) = x(t) - p(t) + t(P_0 O(h^k) + Q_0 O(h^l))$ provided that $P_0 x \in C^{\tilde{k}+1}[0,1]$ or equivalently $Dx \in C^{\tilde{k}+1}[0,1]$ and $Q_0 x \in C^{\tilde{l}+1}[0,1]$, where $k := \min\{\tilde{k}, m\}$ and $l := \min\{\tilde{l}, m\}$. Now, the error e satisfies the collocation scheme

$$A(t_{i,j})De'(t_{i,j}) + B(t_{i,j})e(t_{i,j}) = t_{i,j}B(t_{i,j})(P_0 O(h^k) + Q_0 O(h^l)), \quad e(0) = 0$$

which we again decouple. According to [4] we have $e_{\text{diff}} := De(x) = tO(h^k)$, and we can use this information to estimate $Q_0 e(t)$. Finally, $x(t) - p(t) = O(h^{\min\{l,k\}})$ follows. For details, the reader is referred to [5].

Numerical experiment. Finally, we present some numerical results to illustrate the theory.

Example: For the test problem specified in (2) the algebraic components and the differential components are given by $Q_0 x(t) = (x_2(t), x_2(t))^T$ and $P_0 x(t) = (x_1(t) - x_2(t), 0)^T = (Dx(t), 0)^T$, respectively. Moreover,

$$G_0(t) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \ G_1(t) = \begin{pmatrix} 1 & 1 \\ 1 & t+1 \end{pmatrix}, \ G_1^{-1}(t) = \frac{1}{t}\begin{pmatrix} t+1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The inherent singular IVP has the form $u'(t) - (-4-2t)/t\, u(t) = (7t+5)e^{5t},\ u(0) = 0$ and $u(t) = te^{5t}$. Since $u(t)$, $x_2(t) \in C^\infty[0,1]$, we have $\tilde{k} = \tilde{l} = \infty$, and thus we expect to see the order of convergence m being the stage order of the method. In the table below we display the estimated convergence order for $m = 2$ equidistantly spaced collocation points, left column, and $m = 2$ Gaussian points, right column. The maximum norm of the global error has been calculated at the meshpoints τ_i.

Mesh	Error for x, equidistant coll.			Error for x, Gaussian coll.		
N	error	order	const.	error	order	const.
10	$1.322e+01$			$9.932e+00$		
20	$3.345e+00$	2.0	$1.271e+03$	$2.511e+00$	2.0	$9.572e+02$
40	$8.409e-01$	2.0	$1.307e+03$	$6.310e-01$	2.0	$9.819e+02$
80	$2.110e-01$	2.0	$1.320e+03$	$1.583e-01$	2.0	$9.906e+02$
160	$5.291e-02$	2.0	$1.324e+03$	$3.970e-02$	2.0	$9.936e+02$
320	$1.327e-02$	2.0	$1.326e+03$	$9.954e-03$	2.0	$9.948e+02$

Mesh	Error for u, equidistant coll.			Error for u, Gaussian coll.		
N	error	order	const.	error	order	const.
10	$7.122e-01$			$3.172e-02$		
20	$1.729e-01$	2.0	$7.847e+01$	$2.029e-03$	4.0	$2.937e+02$
40	$4.290e-02$	2.0	$7.152e+01$	$1.275e-04$	4.0	$3.165e+02$
80	$1.070e-02$	2.0	$6.935e+01$	$7.984e-06$	4.0	$3.240e+02$
160	$2.675e-03$	2.0	$6.871e+01$	$4.992e-07$	4.0	$3.262e+02$
320	$6.685e-04$	2.0	$6.853e+01$	$3.120e-08$	4.0	$3.269e+02$

The numerical results are in good agreement with the theory. The superconvergence does not hold in general although it can be observed for the differential components here. However, if we rerun the test for $m = 3$ Gaussian points, we see the $O(h^4)$ convergence for u again, and not the superconvergence behavior $O(h^6)$, see [5].

Conclusion. The concept of a properly stated leading term and the associated decoupling technique are powerful tools which we were able to utilize in the convergence proof of a collocation method applied to approximate solutions of singular DAEs. The results presented here will be subject to generalizations, such as variable matrix D, general spectrum of $M(0)$, nonlinear homogeneity, and more involved types of critical points.

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Stochastic DAEs in circuit simulation

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One of the challenges of the downscaling in the production of electronic chips is the small signal-to-noise-ratio. In several applications the noise influences the system behaviour in an essentially nonlinear way such that linear noise analysis is no longer satisfactory and transient noise analysis, i.e., the integration of noisy