Iterated Defect Correction for the Solution of Singular Initial Value Problems

Othmar Koch
Ewa B. Weinmüller

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Technische Universität Wien
Vienna University of Technology

Institute for Applied Mathematics
and Numerical Analysis

ITERATED DEFECT CORRECTION FOR THE SOLUTION OF SINGULAR INITIAL VALUE PROBLEMS*

OTHMAR KOCH AND EWA B. WEINMÜLLER†

Abstract. We investigate the convergence properties of the Iterated Defect Correction (IDeC) method based on the implicit Euler rule for the solution of singular initial value problems with a singularity of the first kind. We show that the method retains its classical order of convergence which means that the sequence of approximations obtained during the iteration shows gradually growing order of convergence limited by the smoothness of the data and technical details of the procedure.

Key words. Ordinary differential equations, initial value problems, singularity of the first kind, numerical solution, convergence analysis, implicit Euler rule, acceleration techniques, iterated defect correction.

AMS subject classification. 65L05

1. Preliminaries. We are interested in the numerical solution of nonlinear singular initial value problems of the first order,

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \quad t \in (0, 1], \tag{1.1a}$$

$$B_0 z(0) = \beta, \tag{1.1b}$$

$$z \in C[0, 1], \tag{1.1c}$$

where z, f are vector-valued functions of dimension n, M is an $n \times n$ matrix, B_0 is an $r \times n$ matrix and β is a vector of dimension $r \le n$. Analytical properties of the initial value problems (1.1) have been studied in [24] and [25] and for the reader's convenience the most important results are briefly recapitulated in Appendix A. It turns out that the smoothness conditions (1.1c) can be equivalently expressed in the form of a set of $n - r$ homogeneous initial conditions and the set (1.1b) is augmented by these conditions for the numerical treatment.

Mathematical models of various applications from physics, chemistry and mechanics (e. g. Thomas-Fermi differential equation, Ginzburg-Landau equation, problems in shell buckling) take the form of boundary value problems of second order, where the system of differential equations has the form

$$y''(t) = \frac{A_1(t)}{t} y'(t) + \frac{A_0(t)}{t^2} y(t) + g(t, y(t)), \quad t \in (0, 1], \tag{1.2}$$

and A_0, A_1 are $n \times n$ matrices. Note that (1.2) can be reduced to the form (1.1a) using the linear transformation $z(t) = (y(t), ty'(t))$. Hence, an efficient solution technique for (1.1) can also be used to approximate the solution of (1.2) together with a set of $2n$ appropriately defined initial conditions, for details see [24].

Singular problems arise in a broad range of research fields. Problems as different as the solution of differential equations posed on unbounded intervals, see [1], [8] and [34], the computation of connecting orbits or invariant manifolds for dynamical systems, see [31], [32], differential-algebraic equations ([29]) or Sturm-Liouville eigenvalue problems ([6], [28]) are in the scope of techniques for singular boundary value problems. Often,

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†Both: Department of Applied Mathematics and Numerical Analysis, Vienna University of Technology, Austria.

numerical methods are used to approximate the solution, cf. [4], [7], [9], [22]. Initial value problems of the form (1.1) are also encountered in ecology in the computation of avalanche run-up, see [30].

These and other similar activities in related areas are a strong motivation for the search for a numerical method to be used as a basis for a reliable standard code designed especially for the solution of *singular boundary* value problems, and taking into account the specific difficulties caused by singularities. Unfortunately, the standard direct discretization methods are often disadvantageously effected by the singularity. The application of collocation methods to singular first and second order systems has been studied in [19] and [37]. The main advantage of this computationally expensive method is its high super-convergence order which cannot be guaranteed in case of singularity, in general. The standard direct three-point discretization of (1.2) shows for certain classes of smooth problems its classical convergence order, but since for this scheme the asymptotic error expansion does not exist in general, the acceleration techniques cannot be successfully used to increase the order of the method, see [35], [15]. In [21] several modifications of the standard scheme have been proposed. The discretization error now has a uniform h^2 expansion but unfortunately, this expansion holds only under strong restrictions on the class of singular problems to be solved.

Our aim is to develop a shooting procedure, cf. [2] or [23], and therefore we need to propose an efficient numerical solver for the *singular initial* value problems involved. The reason why we choose the indirect approach is not only the unsatisfactory performance of the high order discretization methods applied to solve boundary value problems directly. More importantly, shooting seems to be a particularly attractive alternative, because within its framework one can use different controlling mechanisms close and away from the singular point. In the context of initial value problems, we have already gained some information how the usual strategies for the error estimation and grid selection work when they are applied to singular problems. We first experimentally investigated the quality of the estimation of the local discretization error of implicit Runge-Kutta methods. Typically, we observed that close to the singular point these strategies do not work efficiently in general. Due to the unsmoothness of the local error near the singular point, the estimation routines tend to overestimate it (even for small stepsizes) and consequently, the selected grids are too fine in relation to the smoothness of the solution. On the other hand, the error estimation procedure for the *global discretization error* based on the IDeC idea (basic solution obtained by Runge-Kutta method) performs completely satisfactory along the whole integration interval. These observations suggest that we will possibly need to control the global error close to the singularity and switch to the control of the local error away from the singular point. This idea can be easily realized in a code based on the shooting approach.

For the numerical solution of singular initial value problems, linear multistep methods and explicit Runge-Kutta schemes, see [18] and [20], respectively, have been investigated. It turns out, however, that all these methods show order reductions when the singularity is present. In this paper we propose an alternative high order method to approximate the solution of (1.1) which is based on the acceleration technique known as Iterated Defect Correction, where the implicit Euler rule serves as the basic method. This analysis was motivated by successful numerical experiments reported in [3] and [25]. Moreover, it was shown in [26] that other one-step methods like the box scheme or the trapezoidal rule cannot be used as basic methods for singular problems. Therefore we restrict our attention to the implicit Euler method.

For regular problems, IDeC was proposed by Frank, see for example [11], [12], and its convergence behavior has been discussed in [10]–[14] or [33]. The analysis in [13] shows that under certain conditions the IDeC iterates converge to a collocation solution which is always a fixed point of the iteration. In [10]–[12] the asymptotic behavior of the respective approximations with respect to the discretization parameter h is investigated. This is also the type of analysis which is the aim of this paper. The results presented in the above papers suggest that the proofs of the classical convergence order for the basic method, and the existence of a sufficiently long expansion of its global discretization error are the necessary prerequisites for the convergence of IDeC. These preliminary considerations for the implicit Euler rule applied to (1.1) have been published in [25] and [27] and are briefly discussed in Appendix B.

Throughout the paper, the following notation is used. We denote by $\mathbb{C}^n$ the space of complex-valued vectors of dimension n and use $|\cdot|$,

$$|x| = |(x_1, x_2, \ldots, x_n)| := \max_{1\le i\le n} |x_i|,$$

to denote the maximum norm in $\mathbb{C}^n$. $C_n^p[0,1]$ is the space of complex vector-valued functions which are p times continuously differentiable on $[0,1]$. For every function $y \in C_n^0[0,1]$ we define the maximum norm,

$$\|y\| := \max_{0\le t\le 1} |y(t)|.$$

We will also use the maximum norm restricted to the interval $[0,\delta]$, $0 < \delta \le 1$,

$$\|y\|_\delta := \max_{0\le t\le \delta} |y(t)|.$$

$C_{n\times n}^p[0,1]$ is the space of complex-valued $n \times n$ matrices with columns in $C_n^p[0,1]$. For a matrix $A = (a_{ij})_{i,j=1}^n$, $A \in C_{n\times n}^0[0,1]$, $\|A\|$ is the induced norm,

$$\|A\| := \max_{0\le t\le 1} |A(t)| = \max_{0\le t\le 1} \left(\max_{1\le i\le n} \sum_{j=1}^n |a_{ij}(t)| \right),$$

and $\|A\|_\delta$ is defined in an obvious way. Where there is no confusion we will delete the subscripts n and $n\times n$ and call $C = C[0,1] = C^0[0,1]$.

For a constant matrix A the kernel of A is denoted by $\ker(A)$ and we write I_n for the identity matrix in $\mathbb{R}^{n\times n}$.

For the numerical analysis, we define equidistant grids of the form

$$\Delta_h := (t_0, t_1, \ldots, t_N),$$

where $t_i = ih,\ i = 0,\ldots,N,\ h = \frac{1}{N}$, and grid vectors

$$u_h := (u_0, \ldots, u_N).$$

The norm on the space of grid vectors is given by

$$\|u_h\|_h := \max_{0\le k\le N} |u_k|.$$

For a continuous function $x(t) \in C[0,1]$, we denote by R_h the projection onto the space of grid vectors,

$$R_h(x) := (x(t_0), \ldots, x(t_N)).$$

Often, grids will be used for which $N = mN_1,\ m, N_1 \in \mathbb{N}$. In this case we write $\tau_i := t_{im},\ i = 0,\ldots,N_1,\ J_i := [\tau_i, \tau_{i+1}],\ i = 0,\ldots,N_1-1$.

2. The IDeC method. This acceleration technique is based on Zadunaisky's idea for the estimation of the global discretization error of Runge-Kutta methods, see [38]. In this section we briefly explain how this acceleration technique works; a proof of its convergence properties is given in §3. We consider initial value problems of the form

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \quad t \in (0,1], \tag{2.1a}$$

$$z(0) = \beta \in \ker(M(0)). \tag{2.1b}$$

We assume to know the approximate solution, $z_h^{[0]} := z_h$, obtained by the implicit Euler rule on a grid Δ_h, $h = \frac{1}{mN_1} = \frac{1}{N}$, and denote by $p^{[0]}(t) = p_i^{[0]}(t)$, $t \in J_i$, $i = 0, \ldots, N_1 - 1$, the piecewise polynomial of degree m interpolating the values of $z_h^{[0]}$,

$$p_i^{[0]}(t_j) = z_j^{[0]}, \quad j = im, \ldots, (i+1)m, \; i = 0, \ldots, N_1 - 1.$$

Using this interpolating function, we construct a neighboring problem associated with (2.1) and solved exactly by $p^{[0]}(t)$,

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)) + d^{[0]}(t), \quad t \in (0,1], \tag{2.2a}$$

$$z(0) = p^{[0]}(0) = \beta, \tag{2.2b}$$

where

$$d^{[0]}(t) := p^{[0]'}(t) - \frac{M(t)}{t} p^{[0]}(t) - f(t, p^{[0]}(t)).$$

Note that at the points $\tau_i = t_{im}$, $i = 1, \ldots, N_1 - 1$, the derivative of $p^{[0]}(t)$ has a jump discontinuity in general. This means that the left-sided and the right-sided derivative has to be used alternatively for the definition of the neighboring problem on the subsequent subintervals. Using the Taylor expansion of $p^{[0]}(t)$ at $t = 0$ it can be shown that $d^{[0]}(t)$ is bounded and integrable. Therefore, the techniques from [24] can be applied to prove that (2.2) is well-posed. Clearly, in this case the solution of (2.2) is only piecewise differentiable, in general.

We now solve (2.2) by the same numerical method (implicit Euler rule) and obtain an approximate solution $p_h^{[0]}$ for $p^{[0]}(t)$. This means that for the solution of the neighboring problem (2.2) we know the global error which we can use to estimate the unknown error of the original problem (2.1),

$$\varepsilon_h = R_h(z) - z_h \approx \delta_h^{[0]} := R_h\big(p^{[0]}\big) - p_h^{[0]} = z_h^{[0]} - p_h^{[0]}. \tag{2.3}$$

Zadunaisky gave the following heuristic argument for his method to work: If the values z_h are good approximations for the values of the solution $R_h(z)$ at the grid points, then the function $p^{[0]}(t)$ is a good approximation for the solution $z(t)$ itself. Consequently, the defect $d^{[0]}(t)$ is small and hence the neighboring problem (2.2) and the original problem (2.1) are closely related. This implies that the global error of the solution of (2.2) is closely related to the global error of the solution of (2.1), and therefore the estimate (2.3) shall provide some dependable information about its size.

Having the estimate for the global error of the solution $z_h^{[0]}$ we are able to improve this solution by setting

$$z_h^{[1]} := z_h^{[0]} + \delta_h^{[0]} = z_h^{[0]} + \left(R_h\left(p^{[0]}\right) - p_h^{[0]}\right).$$

We use these values to define a new interpolating function $p^{[1]}(t)$ by requiring $p_i^{[1]}(t_j) = z_j^{[1]}$, $j = im, \dots, (i+1)m$, $i = 0, \dots, N_1 - 1$, and the associated defect reads

$$d^{[1]}(t) := p^{[1]'}(t) - \frac{M(t)}{t} p^{[1]}(t) - f(t, p^{[1]}(t)).$$

Clearly, the next neighboring problem is

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)) + d^{[1]}(t), \quad t \in (0,1], \tag{2.4a}$$

$$z(0) = \beta, \tag{2.4b}$$

and we solve it again by the Euler method to obtain the approximation $p_h^{[1]}$ which is used to correct the basic solution,

$$z_h^{[2]} := z_h^{[0]} + \delta_h^{[1]} = z_h^{[0]} + \left(R_h\left(p^{[1]}\right) - p_h^{[1]}\right).$$

Clearly, this procedure can be iteratively continued.

3. Convergence of the IDeC method. We now prove the following convergence result:

THEOREM 3.1. *Consider the IDeC method based on the implicit Euler rule and on piecewise interpolation with polynomials of degree m for the numerical solution of problem* (2.1). *For the approximations obtained in the course of the iteration,*

$$\|z_h^{[j]} - R_h(z)\|_h = O(h^{j+1}), \quad j = 0, \dots, m-1, \tag{3.1}$$

holds, provided that f and M are sufficiently smooth. In this case (polynomials of degree m are used for the interpolation), further iteration does not increase the asymptotic order of the approximation in general.

Proof. For simplicity, we restrict our attention to two steps of the IDeC. The generalization of the proof is straight-forward. Let us choose $m = 3$[1]. For sufficiently smooth data, it follows from Theorem B.2 that an asymptotic error expansion for the basic solution exists. It has the form (B.2) with $m = 3$. Under the same assumptions, similar expansions exist for the neighboring problems (2.2) and (2.4). Since the neighboring problems depend on h, so do the coefficient functions in the associated error expansions. However, we do not use an additional subscript h to emphasize this dependence in order to keep the notation simple. Nonetheless, it is important to note that the expansion gives an asymptotic relation with respect to a family of step-sizes $\bar{h} = \frac{h}{\nu}$, $\nu \in \mathbb{N}$, for a problem which depends on a fixed parameter h. Later on in the discussion it is possible to choose the special step-size $\bar{h} = h$, but at this point it is convenient to make the difference between problem-dependent influences (h) and

[1] To show how the generalization works, the results will be given for general m whenever this does not result in an unnecessarily complicated notation.

aspects of the numerical method ($\bar{h}$) as clear as possible. Thus, the error expansions read

$$p_{\bar{h}}^{[l]} - R_{\bar{h}}\left(p^{[l]}\right) = \sum_{j=1}^{m} \bar{h}^j R_{\bar{h}}\left(e_j^{[l]}\right) + r_{\bar{h}}^{[l]}, \quad l = 0,1, \tag{3.2}$$

where $e_j^{[l]},\ j = 1,\dots,m$, are continuous functions defined by piecewise variational equations of the form specified in Theorem B.2. These functions may have jump discontinuities in the first derivatives at $\tau_i = t_{im},\ i = 1,\dots,N_1-1$, but in the interior of the intervals $J_i,\ i = 0,\dots,N_1-1$, they have the same smoothness properties as e_j from (B.2). For the remainder term the following estimate holds:

$$\|r_{\bar{h}}^{[l]}\|_{\bar{h}} \le C(h)\bar{h}^{m+1}, \quad l = 0,1, \tag{3.3}$$

where the constant depends on the problem and therefore on h. It is crucial to show that $C(h) \le C$ for sufficiently small $h \le h_0$. We will turn back to this point later in the discussion.
¿From now on, let $\bar{h} := h$. Then from the existence of the error expansions for the original and the neighboring problems we can conclude

$$R_h(z) - z_h^{[l+1]} = \sum_{j=1}^{m} h^j \left(R_h(e_j) - R_h\left(e_j^{[l]}\right)\right) + r_h - r_h^{[l]}, \quad l = 0,1. \tag{3.4}$$

Consequently, the proof is completed by showing that (3.3) and

$$\|R_h(e_j) - R_h\left(e_j^{[l]}\right)\|_h = O(h^{2+l-j}), \quad j = 1,\dots,m,\ l = 0,1, \tag{3.5}$$

hold. We first recursively prove (3.5), $j = 1,\dots,m,\ l = 0,1$, and finally show (3.3). For the estimates of the solution of the variational equations and the associated neighboring problems, we require the stability result of Lemma C.1, see Appendix C.
Moreover, in order to show (3.5), we also require the following relationship between the derivatives of the exact solution of (2.1) and the solution of the first neighboring problem, which holds for sufficiently smooth $f(t,z)$ and $M(t)$:

$$\|p^{[0](k)} - z^{(k)}\| = \begin{cases} O(h), & k = 0,\dots,m = 3, \\ O(1), & k \ge m+1 = 4. \end{cases} \tag{3.6}$$

To prove this assertion, let $r^{[0]}(t)$ be the continuous interpolant of r_h from (B.2) consisting of piecewise polynomials $r_i^{[0]}(t),\ t \in J_i,\ i = 0,\dots,N_1-1$, of degree $m = 3$. Written in the Lagrange form, these polynomials read

$$r_i^{[0]}(t) = \sum_{l=0}^{m} r_{im+l} \frac{\omega_i(t)}{(t - t_{im+l})\omega_i'(t_{im+l})} =: \sum_{l=0}^{m} r_{im+l} L_{im+l}(t),$$

where

$$\omega_i(t) := \prod_{j=0}^{m} (t - t_{im+j}) = O(h^{m+1}),$$
$$\omega_i^{(k)}(t) = O(h^{m+1-k}), \quad k = 1,\dots,m+1.$$

Thus,

$$\begin{aligned} L_{im+l}^{(k)}(t) &= O(h^{-k}), \quad k=0,\dots,m,\\ L_{im+l}^{(k)}(t) &= 0, \quad k=m+1,\dots,\\ r^{[0](k)}(t) &= O(h^{m+1-k}), \quad k=0,\dots,m,\\ r^{[0](k)}(t) &= 0, \quad k=m+1,\dots. \end{aligned}$$

We now consider a continuous interpolant of $z_h^{[0]}$,

$$\Psi^{[0]}(t) := z(t) + \sum_{j=1}^{m} h^j e_j(t) + r^{[0]}(t), \quad t \in [0,1]. \tag{3.7}$$

Obviously, $\Psi^{[0]}(t)$ interpolates the function $p^{[0]}(t)$ at the points $t_j,\ j=0,\dots,N$. Hence, for $t \in J_i,\ i=0,\dots,N_1-1$,

$$p^{[0](k)}(t) - \Psi^{[0](k)}(t) = \sum_{l=0}^{k} \frac{k!}{(m+k-l+1)!l!}\omega_i^{(l)}(t)\Psi^{[0](m+k-l+1)}(\xi_{il}) \tag{3.8}$$

holds for $k=0,\dots,m$ and $\xi_{il} \in J_i,\ l=0,\dots,k$, provided f and M are sufficiently smooth[2], see [16] for a proof. Note that at $\tau_i,\ i=1,\dots,N_1-1$, the statement holds for the right-sided and the left-sided derivatives, respectively.

Finally, (3.6) follows by using $\Psi^{[0]}(t)$ and the triangle inequality in order to estimate $p^{[0](k)}(t) - z^{(k)}(t)$.

We now use Lemma C.1 to derive the estimates (3.5). We first show that the assumptions of this lemma are satisfied by the variational equations and the associated neighboring problems.

For our considerations we choose a sufficiently large domain which contains $z(t), e_1(t), \dots, e_m(t)$, as well as all their required derivatives. Then we show recursively that $p^{[0]}(t), e_1^{[0]}(t), \dots, e_m^{[0]}(t)$, and their derivatives are confined to a bounded domain G. For $p^{[0]}(t)$ this follows from (3.6). For the solution of the first piecewise variational equation we can conclude, cf. [24], that

$$\|e_1^{[0]}\|_\delta \le \text{const}\,\|p^{[0]''}\|_\delta$$

holds for sufficiently small δ. Classical theory now implies that $e_1^{[0]}$ is bounded on $[0,1]$.

Since $\frac{\partial f(t,z)}{\partial z}$ is bounded on the domain $[0,1]\times G$, the function S_1, cf. Theorem B.2, and the analogously defined function $S_1^{[0]}$ associated with the first neigboring problem satisfy a Lipschitz condition. Consequently,

$$\max_{(t,e_1)\in[0,1]\times G} \left| S_1\left(t, e_1, z(t), z''(t)\right) - S_1^{[0]}\left(t, e_1, p^{[0]}(t), p^{[0]''}(t)\right)\right| = O(h)$$

follows. This is just condition (C.3) from Lemma C.1. Thus, the lemma can be used to derive the estimate (3.5) for $j=1,\ l=0$.

In order to estimate $e_1' - e_1^{[0]'}$, we substitute the representation (A.3) of $e_1(t)$,

$$e_1(t) = t\int_0^1 \tau^{-M(0)} S_1(\tau t, e_1(\tau t), z(\tau t), z''(\tau t))\,d\tau,$$

[2] For $k=3$ we require $f \in C^9,\ M \in C^{10}$.

into the right-hand side of the defining variational equation, and likewise for $e_1^{[0]}(t)$. Using the triangle inequality and the fact that the derivatives of f are bounded on the domain $[0,1]\times G$, we obtain

$$
\begin{aligned}
e_1'(t) - e_1^{[0]'}(t) = M(0)\int_0^1 \tau^{-M(0)} \Big(& S_1\left(\tau t, e_1(\tau t), z(\tau t), z''(\tau t)\right) \\
& - S_1^{[0]}\left(\tau t, e_1^{[0]}(\tau t), p^{[0]}(\tau t), p^{[0]''}(\tau t)\right)\Big)\, d\tau \\
& + S_1\left(t, e_1(t), z(t), z''(t)\right) \\
& - S_1^{[0]}\left(t, e_1^{[0]}(t), p^{[0]}(t), p^{[0]''}(t)\right) \\
= O(h). &
\end{aligned} \tag{3.9}
$$

For the higher derivatives of e_1 and $e_1^{[0]}$ we differentiate the equation (3.9) and apply repeatedly the triangle inequality. In these estimates, and likewise for $e_j,\ j=2,\dots,m$, we use the equivalence

$$\prod_{\kappa=1}^{j} x_\kappa - \prod_{\kappa=1}^{j} y_\kappa = \sum_{\mu=1}^{j}\prod_{\nu=1}^{\mu-1} y_\nu (x_\mu - y_\mu) \prod_{\kappa=\mu+1}^{j} x_\kappa$$

which holds for any commutative $x_\kappa, y_\kappa,\ \kappa = 1,\dots,j$. Note that this means that the estimates depend only on the differences between the arguments (and their derivatives) of S_j and $S_j^{[0]}$. These quantities can easily be determined from the definitions in Theorem B.2. Thus, if $f(t,z)$ and $M(t)$ are sufficiently smooth, we obtain[3]

$$
\begin{aligned}
\|e_1^{[0](k)} - e_1^{(k)}\| &= \begin{cases} O(h), & k=0,1,2,\\ O(1), & k=3,4,\end{cases}\\
\|e_2^{[0](k)} - e_2^{(k)}\| &= \begin{cases} O(h), & k=0,1,\\ O(1), & k=2,3,\end{cases}\\
\|e_3^{[0](k)} - e_3^{(k)}\| &= O(1), \quad k=0,1,2.
\end{aligned}
$$

Using the same arguments, we can also prove that the inhomogeneities in the difference equations for the remainder terms satisfy

$$\|l_h - l_h^{[0]}\|_h = O(h^4).$$

Thus it follows from the estimate in Theorem B.2 that the hypothesis (3.3) for $l=0$ holds which completes the first step of the proof,

$$\|z_h^{[1]} - R_h(z)\|_h = O(h^2).$$

For the convergence proof of the second iteration step, we argue in a very similar manner.

[3]In these and the following similar estimates we use $m=3$ explicitly since for general m different cases have to be discerned due to the fact that the results for the first derivatives are always the same as for the original functions. Apart from the resulting awkward notation, the generalization can be done in a straight-forward way by successively substituting previous estimates into the defining variational equations and differentiating as before.

If $r^{[1]}(t) := r_i^{[1]}(t),\ t \in J_i,\ i = 0, \dots, N_1 - 1$ is the continuous interpolant of $r_h - r_h^{[0]} = O(h^{m+1})$ consisting of piecewise polynomials of degree $m = 3$, then

$$\Psi^{[1]}(t) := z(t) + \sum_{j=1}^{m} h^j \left(e_j(t) - e_j^{[0]}(t)\right) + r^{[1]}(t), \quad t \in [0,1],$$

is an interpolant of $z_h^{[1]}$. Since the same holds for $p^{[1]}(t)$ we can conclude from the analogue of (3.8) that

$$\|\Psi^{[1](k)} - p^{[1](k)}\| = \begin{cases} O(h^{4-k}), & k = 0, \dots, 3, \\ O(1), & k = 4, 5, \end{cases}$$

holds if $\Psi^{[1]}(t)$ is sufficiently smooth. Moreover, using the same interpolation arguments for $r^{[1]}(t)$ as for $r^{[0]}(t)$ before and taking into account the previous estimates for $\|e_j^{[0](k)} - e_j^{(k)}\|$, $j = 1, 2, 3$, it follows from the definition of $\Psi^{[1]}(t)$ that

$$\|\Psi^{[1](k)} - z^{(k)}\| = \begin{cases} O(h^2), & k = 0, 1, 2, \\ O(h), & k = 3, \\ O(1), & k = 4, 5. \end{cases}$$

These estimates are again derived by using the appropriate estimates for arguments of S_j and $S_j^{[1]}$, respectively, by substituting the results into the differential equation as in (3.9) and differentiating the resulting relations.
Now the following estimates follow for sufficiently smooth $f(t, z)$ and $M(t)$:

$$\|p^{[1](k)} - z^{(k)}\| = \begin{cases} O(h^2), & k = 0, 1, 2, \\ O(h), & k = 3, \\ O(1), & k = 4, 5, \end{cases}$$

$$\|e_1^{[1](k)} - e_1^{(k)}\| = \begin{cases} O(h^2), & k = 0, 1, \\ O(h), & k = 2, \\ O(1), & k = 3, 4, \end{cases}$$

$$\|e_2^{[1](k)} - e_2^{(k)}\| = \begin{cases} O(h), & k = 0, 1, \\ O(1), & k = 2, 3, \end{cases}$$

$$\|e_3^{[1](k)} - e_3^{(k)}\| = O(1), \quad k = 0, 1, 2.$$

Finally, an analogous argument for the remainder term completes the proof of Theorem 3.1 for the case $m = 3$. In the general case the proof can be carried out in a very similar way, using

$$\Psi^{[l-1]}(t) := z(t) + \sum_{j=1}^{l+1} h^j \left(e_j(t) - e_j^{[l-2]}(t)\right) + r^{[l-1]}(t), \quad t \in [0,1],$$

as an interpolating function in the l-th step. □

Remark: An approach to prove the result for a general step of the IDeC iteration that is different from the one adopted here is given in [14]. There, the result of the $(m-1)$-st step is considered as the result of an arbitrary numerical method and one additional IDeC step is performed. However, we believe that the direct approach presented here provides a clear indication of how the argument can be continued.

4. Numerical Results. To illustrate the theory we consider a nonlinear problem from [20],

$$y''(t) = -\frac{1}{t}y'(t) + y^3(t) - 3y^5(t), \quad t \in (0,1],$$
$$y(0) = 1, \; y'(0) = 0.$$

The equivalent first order form has been used for the computation,

$$v'(t) = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v(t) + t\begin{pmatrix} 0 \\ v_1^3(t) - 3v_1^5(t) \end{pmatrix}, \quad t \in (0,1],$$
$$v(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v'(0) = 0.$$

The exact solution of this problem reads $y(t) = \frac{1}{\sqrt{1+t^2}}$.
In Table 4.1 the maximal error of the approximation obtained in the j-th step of the IDeC iteration is denoted by δ_j. Moreover, p_j and c_j are the empirical orders of convergence and error constants, respectively. Polynomials of degree five were used for the interpolation. In accordance with the theory, Table 4.1 shows the expected order sequence $O(h), O(h^2), \dots, O(h^5)$, with no further improvement after the fourth iteration.

Appendix A. Analytical Results for Singular Initial Value Problems. Consider the singular initial value problem of the first order

$$z'(t) = \frac{M(t)}{t}z(t) + f(t, z(t)), \quad t \in (0,1], \tag{A.1a}$$
$$B_0 z(0) = \beta, \tag{A.1b}$$
$$M(0)z(0) = 0, \tag{A.1c}$$

where z, f are n-dimensional vector-valued functions, M is an $n \times n$ matrix, B_0 is an $r \times n$ matrix and β is an r-dimensional vector, $r \le n$. We assume $M(t) \in C^1[0,1]$ which means that we can write

$$M(t) = M(0) + tC(t) \tag{A.2}$$

with $C(t) \in C[0,1]$. For the numerical treatment we always assume $B_0 = I_n$ and $\beta \in \ker(M(0))$. For suitable $\beta = \tilde{E}\beta_r$, where $\tilde{E}$ is a basis of the kernel of $M(0)$, this choice provides r linearly independent initial conditions equivalent to (A.1b).
The analytical properties of (A.1) have been investigated in full detail in [24] and in [25]. For the analysis we assume that $f(t,z)$ is continuous and satisfies a Lipschitz-condition with respect to z on $[0,1] \times \mathbb{R}^n$. It can be shown that a restriction on the spectrum of $M(0)$, namely the absence of purely imaginary eigenvalues or eigenvalues with positive real parts, is necessary in order to formulate an initial value problem of the form (A.1) having a unique, continuous solution $y(t)$. In this case, the condition $M(0)z(0) = 0$ is necessary and sufficient for $z \in C[0,1]$ and provides $n - r$ linearly independent conditions which the initial value $z(0)$ has to satisfy. This solution is unique iff the $r{\times}r$ matrix $B_0\tilde{E}$ is nonsingular. If f is k times continuously differentiable and $M \in C^{k+1}[0,1]$, then the solution satisfies $z \in C^{k+1}[0,1]$. The solution of (A.1) is the unique solution of the equivalent integral equation

$$z(t) = z(0) + t\int_0^1 \tau^{-M(0)}C(\tau t)z(\tau t)\,d\tau + t\int_0^1 \tau^{-M(0)} f(\tau t, z(\tau t))\,d\tau. \tag{A.3}$$

TABLE 4.1
Convergence of the IDeC method on $[0,1]$.

h	δ_0	p_0	c_0	δ_1	p_1	c_1
$1/5$	$7.9\cdot10^{-02}$	0.565	$-1.9\cdot10^{-01}$	$5.3\cdot10^{-02}$	1.334	$-4.5\cdot10^{-01}$
$1/5\cdot2^{-1}$	$5.3\cdot10^{-02}$	0.761	$-3.1\cdot10^{-01}$	$2.1\cdot10^{-02}$	1.661	$-9.6\cdot10^{-01}$
$1/5\cdot2^{-2}$	$3.1\cdot10^{-02}$	0.875	$-4.3\cdot10^{-01}$	$6.6\cdot10^{-03}$	1.830	$-1.6\cdot10^{+00}$
$1/5\cdot2^{-3}$	$1.7\cdot10^{-02}$	0.936	$-5.4\cdot10^{-01}$	$1.8\cdot10^{-03}$	1.915	$-2.1\cdot10^{+00}$
$1/5\cdot2^{-4}$	$9.0\cdot10^{-03}$	0.968	$-6.2\cdot10^{-01}$	$4.9\cdot10^{-04}$	1.957	$-2.6\cdot10^{+00}$
$1/5\cdot2^{-5}$	$4.6\cdot10^{-03}$	0.983	$-6.7\cdot10^{-01}$	$1.2\cdot10^{-04}$	1.978	$-2.9\cdot10^{+00}$
$1/5\cdot2^{-6}$	$2.3\cdot10^{-03}$	0.991	$-7.1\cdot10^{-01}$	$3.2\cdot10^{-05}$	1.989	$-3.1\cdot10^{+00}$
$1/5\cdot2^{-7}$	$1.1\cdot10^{-03}$	0.995	$-7.3\cdot10^{-01}$	$8.1\cdot10^{-06}$	1.994	$-3.2\cdot10^{+00}$
$1/5\cdot2^{-8}$	$5.8\cdot10^{-04}$	0.997	$-7.4\cdot10^{-01}$	$2.0\cdot10^{-06}$	1.997	$-3.3\cdot10^{+00}$
$1/5\cdot2^{-9}$	$2.9\cdot10^{-04}$	0.998	$-7.4\cdot10^{-01}$	$5.1\cdot10^{-07}$	1.998	$-3.3\cdot10^{+00}$
$1/5\cdot2^{-10}$	$1.4\cdot10^{-04}$	0.999	$-7.5\cdot10^{-01}$	$1.2\cdot10^{-07}$	1.999	$-3.3\cdot10^{+00}$
$1/5\cdot2^{-11}$	$7.3\cdot10^{-05}$	0.999	$-7.5\cdot10^{-01}$	$3.2\cdot10^{-08}$	1.999	$-3.3\cdot10^{+00}$
$1/5\cdot2^{-12}$	$3.6\cdot10^{-05}$	0.999	$-7.5\cdot10^{-01}$	$8.0\cdot10^{-09}$	1.999	$-3.3\cdot10^{+00}$
$1/5\cdot2^{-13}$	$1.8\cdot10^{-05}$	0.999	$-7.5\cdot10^{-01}$	$2.0\cdot10^{-09}$	1.999	$-3.3\cdot10^{+00}$
$1/5\cdot2^{-14}$	$9.2\cdot10^{-06}$			$5.0\cdot10^{-10}$		

h	δ_2	p_2	c_2	δ_3	p_3	c_3
$1/5$	$3.0\cdot10^{-02}$	2.486	$-1.6\cdot10^{+00}$	$1.1\cdot10^{-02}$	2.988	$-1.4\cdot10^{+00}$
$1/5\cdot2^{-1}$	$5.4\cdot10^{-03}$	2.992	$-5.3\cdot10^{+00}$	$1.4\cdot10^{-03}$	2.931	$-1.2\cdot10^{+00}$
$1/5\cdot2^{-2}$	$6.8\cdot10^{-04}$	3.158	$-8.7\cdot10^{+00}$	$1.9\cdot10^{-04}$	3.532	$-7.7\cdot10^{+00}$
$1/5\cdot2^{-3}$	$7.6\cdot10^{-05}$	3.137	$-8.0\cdot10^{+00}$	$1.6\cdot10^{-05}$	3.809	$-2.1\cdot10^{+01}$
$1/5\cdot2^{-4}$	$8.6\cdot10^{-06}$	3.013	$-4.7\cdot10^{+00}$	$1.2\cdot10^{-06}$	3.915	$-3.4\cdot10^{+01}$
$1/5\cdot2^{-5}$	$1.0\cdot10^{-06}$	2.932	$-3.1\cdot10^{+00}$	$8.0\cdot10^{-08}$	3.960	$-4.3\cdot10^{+01}$
$1/5\cdot2^{-6}$	$1.4\cdot10^{-07}$	2.966	$-3.8\cdot10^{+00}$	$5.1\cdot10^{-09}$	3.980	$-4.8\cdot10^{+01}$
$1/5\cdot2^{-7}$	$1.7\cdot10^{-08}$	2.983	$-4.2\cdot10^{+00}$	$3.2\cdot10^{-10}$	3.990	$-5.1\cdot10^{+01}$
$1/5\cdot2^{-8}$	$2.2\cdot10^{-09}$	2.991	$-4.4\cdot10^{+00}$	$2.0\cdot10^{-11}$	3.995	$-5.3\cdot10^{+01}$
$1/5\cdot2^{-9}$	$2.8\cdot10^{-10}$	2.995	$-4.6\cdot10^{+00}$	$1.2\cdot10^{-12}$	3.997	$-5.4\cdot10^{+01}$
$1/5\cdot2^{-10}$	$3.5\cdot10^{-11}$	2.997	$-4.7\cdot10^{+00}$	$8.0\cdot10^{-14}$	3.998	$-5.4\cdot10^{+01}$
$1/5\cdot2^{-11}$	$4.4\cdot10^{-12}$	2.999	$-4.7\cdot10^{+00}$	$5.0\cdot10^{-15}$	3.999	$-5.5\cdot10^{+01}$
$1/5\cdot2^{-12}$	$5.6\cdot10^{-13}$	2.999	$-4.7\cdot10^{+00}$	$3.1\cdot10^{-16}$	3.999	$-5.5\cdot10^{+01}$
$1/5\cdot2^{-13}$	$7.0\cdot10^{-14}$			$1.9\cdot10^{-17}$		

h	δ_4	p_4	c_4	δ_5	p_5	c_5
$1/5$	$7.1\cdot10^{-03}$	2.033	$-1.8\cdot10^{-01}$	$1.2\cdot10^{-02}$	3.227	$-2.1\cdot10^{+00}$
$1/5\cdot2^{-1}$	$1.7\cdot10^{-03}$	4.803	$-1.1\cdot10^{+02}$	$1.2\cdot10^{-03}$	6.104	$-1.6\cdot10^{+03}$
$1/5\cdot2^{-2}$	$6.2\cdot10^{-05}$	5.198	$-3.6\cdot10^{+02}$	$1.8\cdot10^{-05}$	4.802	$-3.3\cdot10^{+01}$
$1/5\cdot2^{-3}$	$1.7\cdot10^{-06}$	5.140	$-2.9\cdot10^{+02}$	$6.6\cdot10^{-07}$	5.269	$-1.8\cdot10^{+02}$
$1/5\cdot2^{-4}$	$4.8\cdot10^{-08}$	5.079	$-2.2\cdot10^{+02}$	$1.7\cdot10^{-08}$	5.324	$-2.3\cdot10^{+02}$
$1/5\cdot2^{-5}$	$1.4\cdot10^{-09}$	5.041	$-1.8\cdot10^{+02}$	$4.3\cdot10^{-10}$	5.250	$-1.6\cdot10^{+02}$
$1/5\cdot2^{-6}$	$4.3\cdot10^{-11}$	5.020	$-1.6\cdot10^{+02}$	$1.1\cdot10^{-11}$	5.165	$-9.9\cdot10^{+01}$
$1/5\cdot2^{-7}$	$1.3\cdot10^{-12}$	5.009	$-1.5\cdot10^{+02}$	$3.1\cdot10^{-13}$	5.099	$-6.4\cdot10^{+01}$
$1/5\cdot2^{-8}$	$4.1\cdot10^{-14}$	4.990	$-1.3\cdot10^{+02}$	$9.2\cdot10^{-15}$	5.056	$-4.7\cdot10^{+01}$
$1/5\cdot2^{-9}$	$1.3\cdot10^{-15}$	4.995	$-1.3\cdot10^{+02}$	$2.7\cdot10^{-16}$	5.031	$-3.9\cdot10^{+01}$
$1/5\cdot2^{-10}$	$4.1\cdot10^{-17}$	4.997	$-1.4\cdot10^{+02}$	$8.4\cdot10^{-18}$	5.016	$-3.4\cdot10^{+01}$
$1/5\cdot2^{-11}$	$1.2\cdot10^{-18}$	4.999	$-1.4\cdot10^{+02}$	$2.6\cdot10^{-19}$	5.009	$-3.2\cdot10^{+01}$
$1/5\cdot2^{-12}$	$4.0\cdot10^{-20}$			$8.1\cdot10^{-21}$		

For sufficiently small $\delta > 0$ the right-hand side of this equation is a contraction for $t \in [0,\delta]$.

Appendix B. The Implicit Euler Rule for Singular Problems. For a grid vector x_h on an equidistant grid Δ_h define the operator

$$\text{(B.1)}\quad F_h(x_h) := \begin{pmatrix} \frac{x_{i+1}-x_i}{h} - \frac{M(t_{i+1})}{t_{i+1}} x_{i+1} - f(t_{i+1}, x_{i+1}), \;\; i = 0,\dots,N-1 \\ x_0 - \beta \end{pmatrix}.$$

Clearly, the approximation z_h for the solution of (A.1) obtained by the implicit Euler

rule solves the nonlinear scheme $F_h(z_h) = 0$.
In [25] and [27] a contraction argument on a subgrid on the interval $[0, \delta]$ and the classical result on $[\delta, 1]$ were used to show that the following convergence result holds:
THEOREM B.1. *Consider the system* (B.1) *with the nonlinear operator* $F_h(z_h)$. *For every continuous* $f(t, z)$ *which has a bounded Fréchet-derivative with respect to* z *on* $[0,1] \times \mathbb{R}^n$, $M(t) \in C^1[0,1]$ *and every initial value* $\beta \in \ker(M(0))$, *the nonlinear scheme* $F_h(z_h) = 0$ *has a unique solution* z_h. *If the solution* z *of the underlying analytical problem* (A.1) *satisfies* $z \in C^2[0,1]$, *then the global error* $\varepsilon_h := R_h(z) - z_h$ *of the approximate solution satisfies*

$$\|\varepsilon_h\|_h = O(h), \quad h \to 0.$$

Moreover, Newton's method applied to solve the nonlinear problem (B.1) converges quadratically, provided that f is smooth, see [27] for the proof.
In addition, an arbitrarily long asymptotic error expansion for z_h exists for sufficiently smooth problem data $f(t, z)$ and $M(t)$. Examples for such expansions can be found in [25] and [27]. The general form is discussed in the next theorem.
THEOREM B.2. *Consider* z_h *from the previous theorem as an approximation for the solution of* (A.1). *Let* $M(t) \in C^{m+2}[0,1]$. *Let* $f(t, z)$ *be* $m+1$ *times continuously differentiable on* $[0,1] \times \mathbb{R}^n$ *with bounded derivatives with respect to* z. *Then, there exists an asymptotic error expansion for the global error of* z_h,

$$z_h - R_h(z) = \sum_{j=1}^{m} h^j R_h(e_j) + r_h, \tag{B.2}$$

where $e_j \in C^{m+2-j}[0,1]$, $j = 1, \dots, m$, *are smooth functions which satisfy the variational equations*

$$\begin{aligned}
e_j'(t) &= \frac{M(t)}{t} e_j(t) + \frac{\partial f(t,z)}{\partial z}(t, z(t)) e_j(t) \\
&\quad + \sum_{\mu=2}^{j} \frac{\partial^\mu f(t,z)}{\partial z^\mu}(t, z(t)) \sum_{\substack{\mu_1 + \dots + \mu_j = \mu \\ \mu_i \geq 0,\ i = 1, \dots, j \\ \sum_{\nu=1}^{j} \nu \mu_\nu = j}} \frac{1}{\mu_1! \cdots \mu_j!} \prod_{\kappa=1}^{j} e_\kappa^{\mu_\kappa}(t) \\
&\quad + \sum_{l=1}^{j-1} \frac{(-1)^{j+1-l}}{(j+1-l)!} e_l^{(j+1-l)}(t) + \frac{(-1)^{j+1}}{(j+1)!} z^{(j+1)}(t) \\
&=: \frac{M(0)}{t} e_j(t) + S_j\left(t, e_1, \dots, e_1^{(j)}, \dots, e_{j-1}, \dots, e_{j-1}'', e_j, z, z^{(j+1)}\right), \\
e_j(0) &= 0.
\end{aligned}$$

The remainder term r_h *satisfies the recursion*

$$\begin{aligned}
\frac{r_{i+1} - r_i}{h} &= \frac{M(t_{i+1})}{t_{i+1}} r_{i+1} + g(t_{i+1}, r_{i+1}) + l_i, \quad i = 0, \dots, N-1, \\
r_0 &= 0
\end{aligned}$$

with

$$g(t_i, r_i) := \int_0^1 \frac{\partial f(t,z)}{\partial z}\left(t_i, z(t_i) + \sum_{j=1}^{m} h^j e_j(t_i) + \tau r_i\right) d\tau \cdot r_i,$$

$$l_i := h^{m+1} \sum_{l=1}^{m} \frac{(-1)^{m-l}}{(m+1-l)!} \int_0^1 e_l^{(m+2-l)}(t_{i+1} - \tau h)(1-\tau)^{m+1-l}\, d\tau$$
$$+ h^{m+1} \frac{(-1)^m}{(m+1)!} \int_0^1 z^{(m+2)}(t_{i+1} - \tau h)(1-\tau)^{m+1}\, d\tau$$
$$+ \sum_{\mu=1}^{m} \frac{\partial^\mu f(t,z)}{\partial z^\mu}(t_{i+1}, z(t_{i+1})) \sum_{\substack{\mu_1 + \cdots + \mu_m = \mu \\ \mu_i \geq 0,\ i = 1,\ldots,m \\ \sum_{\nu=1}^m \nu\mu_\nu > m}} \frac{1}{\mu_1! \cdots \mu_m!} \prod_{\kappa=1}^{m} e_\kappa^{\mu_\kappa}(t_{i+1}) h^{\kappa\mu_\kappa}.$$

Moreover, the estimates $\|l_h\|_h = O(h^{m+1})$ *and consequently,*

$$\|r_h\|_h = O(h^{m+1})$$

hold.

Proof. The above variational equations and the difference equation can be derived by substituting the ansatz (B.2) into the defining equation $F_h(z_h) = 0$ (cf. (B.1)), Taylor expansion of the involved quantities and equating coefficients of the same powers of h. The smoothness properties follow by applying the results of Appendix A, thereby justifying the formal Taylor expansions, and the estimate for l_h is straight-forward. The estimate for r_h follows by applying techniques used in the proof of Theorem B.1. The detailed proofs of case studies, $m = 5$ and $m = 8$, can be found in [25] and [27], respectively. □

Remark: We chose to derive the variational equations without using Faa di Bruno's formula because it is not possible to rewrite the singular problem in an autonomous form and use the result directly. Moreover, from the explicit derivation of the variational equations it is immediately clear that their problem type is indeed the same as for the original problem and all previously developed techniques for singular problems can be utilized.

Appendix C. A Stability Result for Piecewise, Singular Problems.

LEMMA C.1. *Consider the singular initial value problem*[4]

$$z'(t) = \frac{M}{t} z(t) + f(t, z(t)), \quad t \in (0,1], \tag{C.1a}$$

$$z(0) = \beta, \tag{C.1b}$$

and the problem

$$y_i'(t) = \frac{M}{t} y_i(t) + g_i(t, y_i(t)), \quad t \in J_i,\ i = 0, \ldots, N_1 - 1, \tag{C.2a}$$

$$y_0(0) = \gamma, \tag{C.2b}$$

$$y_i(\tau_i) = y_{i-1}(\tau_i), \quad i = 1, \ldots, N_1 - 1, \tag{C.2c}$$

[4]The case of variable coefficient matrix $M(t)$ is included in this problem formulation.

with data specified in a piecewise manner and the solution $y(t) := y_i(t),\ t \in J_i,\ i = 0,\dots,N_1-1$. *Let us assume* $z(t)$ *and* $y(t)$ *to be confined to a domain* $G \subseteq \mathbb{R}^n$. *Let* $f(t,z)$ *be continuous, and Lipschitz-continuous with respect to* z *with Lipschitz-constant* L *on* $[0,1]\times G$. *Let the same hold for* $g_i(t,z)$ *on* $J_i \times G$. *Finally, assume that*

$$\max_{(t,z)\in J_i\times G} |f(t,z)-g_i(t,z)| \le \varepsilon, \quad i=0,\dots,N_1-1. \tag{C.3}$$

Then

$$\|z-y\| \le S\mathrm{e}^{L_2} + \frac{\varepsilon}{L_2}\left(\mathrm{e}^{L_2}-1\right) \tag{C.4}$$

if δ *is chosen in such a way that* $L_1 := \delta D L < 1$, *where*

$$\int_0^1 \left|\tau^{-M}\right| \, d\tau \le D = \text{const}, \tag{C.5}$$

$L_2 := 2\max\left\{L, \frac{|M|}{\delta}\right\}$ *and* $S := \frac{1}{1-L_1}|\beta-\gamma| + \frac{\delta D}{1-L_1}\varepsilon$.

Proof. The following representation follows immediately from the variation of constant formula and can also be obtained from (A.3) by the change of variable $\tau \to \frac{s}{t}$:

$$y_i(t) = \gamma + t^M \sum_{\nu=0}^{i-1} \int_{\tau_\nu}^{\tau_{\nu+1}} s^{-M} g_\nu(s, y_\nu(s))\, ds + t^M \int_{\tau_i}^{t} s^{-M} g_i(s, y_i(s))\, ds,$$

$i = 0,\dots,N_1-1$. Likewise,

$$z(t) = \beta + t^M \int_0^t s^{-M} f(s, z(s))\, ds.$$

Combining the above representations for z and y_i, we obtain

$$\begin{aligned}
|z(t)-y_i(t)| \le |\beta-\gamma| &+ \sum_{\nu=0}^{i-1} \int_{\tau_\nu}^{\tau_{\nu+1}} \left|\left(\frac{t}{s}\right)^M\right| ds \cdot \varepsilon \\
&+ L \sum_{\nu=0}^{i-1} \int_{\tau_\nu}^{\tau_{\nu+1}} \left|\left(\frac{t}{s}\right)^M\right| ds \cdot \max_{\eta\in[\tau_\nu,\tau_{\nu+1}]} |z(\eta)-y_\nu(\eta)| \\
&+ \int_{\tau_i}^{t} \left|\left(\frac{t}{s}\right)^M\right| ds \cdot \varepsilon + L \int_{\tau_i}^{t} \left|\left(\frac{t}{s}\right)^M\right| ds \cdot \max_{\eta\in[\tau_i,t]} |z(\eta)-y_i(\eta)| \\
\le |\beta-\gamma| &+ t\int_0^1 \left|\tau^{-M}\right| d\tau \cdot \varepsilon + Lt \int_0^1 \left|\tau^{-M}\right| d\tau \cdot \max_{\eta\in[0,t]} |z(\eta)-y(\eta)|.
\end{aligned}$$

Let δ be sufficiently small and take the maximum over $t \in [0,\delta]$ in the above inequality. Then (C.5) implies

$$\|z-y\|_\delta \le S.$$

Note that this estimate is independent of i and therefore independent of the partition $(\tau_0,\dots,\tau_{N_1})$ involved in the definition of the piecewise problem (C.2).
Now on $[\delta,1]$, the estimate (C.4) is a standard result for regular initial value problems. The extension of the result to piecewise problems of the form (C.2) is straight-forward, for a proof see [10]. ∎

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